

# Generalized Unitarity and Reciprocity Relations for $\mathcal{PT}$ -symmetric Scattering Potentials

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## Abstract

We derive certain identities satisfied by the left/right-reflection and transmission amplitudes,  $R^{l/r}(k)$  and  $T(k)$ , of general  $\mathcal{PT}$ -symmetric scattering potentials. We use these identities to give a general proof of the relations,  $|T(-k)| = |T(k)|$  and  $|R^r(-k)| = |R^l(k)|$ , conjectured in [Z. Ahmed, J. Phys. A **45** (2012) 032004], establish the generalized unitarity relation:  $R^{l/r}(k)R^{l/r}(-k) + |T(k)|^2 = 1$ , and show that it is a common property of both real and complex  $\mathcal{PT}$ -symmetric potentials. The same holds for  $T(-k) = T(k)^*$  and  $|R^r(-k)| = |R^l(k)|$ .

Recently there has been a growing interest in the scattering properties of complex potentials in general and  $\mathcal{PT}$ -symmetric potentials in particular. This is especially motivated by the fact that these potentials are capable of supporting spectral singularities [1, 2, 3, 4, 5] and unidirectional reflectionlessness and invisibility [6, 7]. Spectral singularities [1] correspond to real poles of the left/right-reflection and transmission amplitudes,  $R^{l/r}(k)$  and  $T(k)$ . They are of particular interest in optics, because they correspond to lasing at the threshold gain [3] while their time-reversal yields coherent perfect absorption (CPA) of the electromagnetic radiation [8, 9]. Unidirectional reflectionlessness is characterized by the condition that  $R^{l/r}(k) = 0 \neq R^{r/l}(k)$ , and unidirectional invisibility is described through this condition together with the requirement of perfect transparency:  $T(k) = 1$ . These phenomena are also of great interest, because they provide an interesting root towards designing one-way optical devices [10].

Real scattering potentials are incapable of supporting spectral singularities, CPA, and unidirectional reflectionlessness and invisibility, because they are constrained by the reciprocity and unitarity relations:

$$|R^l(k)| = |R^r(k)|, \quad (1)$$

$$|R^{l/r}(k)|^2 + |T(k)|^2 = 1. \quad (2)$$

While the first of these equations forbids unidirectional reflectionlessness (and hence invisibility), the second identifies the reflection and transmission coefficients,  $|R^{l/r}(k)|^2$  and  $|T(k)|^2$ , with a pair of bounded functions that cannot possess singularities for real wavenumbers  $k$ .

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Complex  $\mathcal{PT}$ -symmetric scattering potentials  $v(x)$ , that by definition satisfy  $v(-x)^* = v(x)$ , can violate one or both of Eqs. (1) and (2). Therefore, they provide a large class of models capable of displaying spectral singularities, CPA, and unidirectional reflectionlessness and invisibility. The fact that the spectral singularities of  $\mathcal{PT}$ -symmetric potentials accompany their time-reversal dual [4] and that they possess the same symmetry as the equations governing unidirectional reflectionlessness and invisibility [7] have made them into a topic of intensive research.

In Ref. [11] the authors derive the following  $\mathcal{PT}$ -symmetric analog of the unitarity condition (2).

$$|T(k)|^2 \pm |R^l(k)R^r(k)| = 1, \quad (3)$$

where  $\pm$  is to be identified with the sign of  $1 - |T(k)|^2$ . This relation is most conveniently established using the properties of the transfer matrix [4]. This is the unique  $2 \times 2$  complex matrix  $\mathbf{M}(k)$  that connects the coefficients of the asymptotic solutions of the wave equation,

$$\psi(x) \rightarrow A_{\pm}(k)e^{ikx} + B_{\pm}(k)e^{-ikx} \quad \text{for } x \rightarrow \pm\infty, \quad (4)$$

according to

$$\begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix} = \mathbf{M}(k) \begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix}. \quad (5)$$

The entries of  $\mathbf{M}(k)$  are related to the reflection and transmission amplitudes via [1]:

$$M_{11}(k) = T(k) - \frac{R^l(k)R^r(k)}{T(k)}, \quad M_{12}(k) = \frac{R^r(k)}{T(k)}, \quad M_{21}(k) = -\frac{R^l(k)}{T(k)}, \quad M_{22}(k) = \frac{1}{T(k)}. \quad (6)$$

In Ref. [5], the author conjectures that the following identities hold for every  $\mathcal{PT}$ -symmetric potential

$$|R^l(-k)| = |R^r(k)|, \quad (7)$$

$$|T(-k)| = |T(k)|, \quad (8)$$

and uses a complexified Scarf II potential to provide evidence for this claim. The present investigation is motivated by the search for a general proof of these relations.

We begin our analysis by recalling that under the action of the space-reflection (parity)  $\mathcal{P}$  and time-reversal transformation  $\mathcal{T}$ , the solutions of the wave equation  $\psi(x)$  transform according to

$$\psi(x) \xrightarrow{\mathcal{P}} \psi(-x), \quad \psi(x) \xrightarrow{\mathcal{T}} \psi(x)^*. \quad (9)$$

It is not difficult to show that these equations together with (4) and (5) imply the following transformation rules for the transfer matrix [9, 4].

$$\mathbf{M}(k) \xrightarrow{\mathcal{P}} \sigma_1 \mathbf{M}(k)^{-1} \sigma_1, \quad (10)$$

$$\mathbf{M}(k) \xrightarrow{\mathcal{T}} \sigma_1 \mathbf{M}(k)^* \sigma_1, \quad (11)$$

$$\mathbf{M}(k) \xrightarrow{\mathcal{PT}} \mathbf{M}(k)^{-1*}, \quad (12)$$

where  $k$  is taken to be real and  $\sigma_1$  is the first Pauli matrix (the  $2 \times 2$  matrix with vanishing diagonal entries and unit off-diagonal entries.) In terms of the entries of  $\mathbf{M}(k)$ , these respectively take the

form

$$M_{11}(k) \xrightarrow{\mathcal{P}} M_{11}(k), \quad M_{12}(k) \xleftrightarrow{\mathcal{P}} -M_{21}(k), \quad M_{22}(k) \xrightarrow{\mathcal{P}} M_{22}(k), \quad (13)$$

$$M_{11}(k) \xleftrightarrow{\mathcal{T}} M_{22}(k)^*, \quad M_{12}(k) \xleftrightarrow{\mathcal{T}} M_{21}(k)^*, \quad (14)$$

$$M_{11}(k) \xleftrightarrow{\mathcal{PT}} M_{22}(k)^*, \quad M_{12}(k) \xleftrightarrow{\mathcal{PT}} -M_{12}(k)^*, \quad M_{21}(k) \xleftrightarrow{\mathcal{PT}} -M_{21}(k)^*. \quad (15)$$

The following are consequences of these relations.

- If  $v(x)$  is an even (real or complex) potential,  $\mathbf{M}$  in  $\mathcal{P}$ -invariant, and (13) implies  $M_{12}(k) = -M_{21}(k)$ . In view of (6), this is equivalent to  $R^l(k) = R^r(k)$ .
- If  $v(x)$  is a real potential,  $\mathbf{M}$  is  $\mathcal{T}$ -invariant, and we find  $M_{11}(k)^* = M_{22}(k)$  and  $M_{12}(k)^* = M_{21}(k)$ . Substituting (6) in these relations leads to (2) and

$$R^l(k)^* = -\frac{R^r(k)T(k)^*}{T(k)}. \quad (16)$$

Let  $\lambda$ ,  $\rho$ , and  $\tau$  be respectively the phase angles of  $R^l(k)$ ,  $R^r(k)$ , and  $T(k)$ , so that

$$R^l(k) = e^{i\lambda(k)}|R^l(k)|, \quad R^r(k) = e^{i\rho(k)}|R^r(k)|, \quad T(k) = e^{i\tau(k)}|T(k)|. \quad (17)$$

Then, we can write (16) as

$$R^l(k)^* = -e^{-2i\tau(k)}R^r(k). \quad (18)$$

Taking the absolute-value of both sides of this relation gives (1). Now, suppose that  $v(x)$  is not reflectionless for the wavenumber  $k$ . Then using (1) and (17) in (18), we find

$$\lambda(k) + \rho(k) = 2\tau(k) + (2m+1)\pi, \quad (19)$$

where  $m$  is an integer.

- If  $v(x)$  is a  $\mathcal{PT}$ -symmetric potential,  $M_{11}(k)^* = M_{22}(k)$  and both  $M_{12}(k)$  and  $M_{21}(k)$  are purely imaginary. In view of (6) and (17), these imply the existence of integers  $m_1$  and  $m_2$  such that [11]

$$\lambda(k) = \tau(k) + \pi(m_1 + \frac{1}{2}), \quad (20)$$

$$\rho(k) = \tau(k) + \pi(m_2 + \frac{1}{2}), \quad (21)$$

$$|T(k)|^2 + e^{i\pi(m_1+m_2)}|R^r(k)R^l(k)| = 1. \quad (22)$$

Clearly (22) coincides with (3). Another interesting observation is that adding Eqs. (20) and (21) side by side gives

$$\lambda(k) + \rho(k) = 2\tau(k) + (m_1 + m_2 + 1)\pi. \quad (23)$$

This coincides with (19) whenever  $m_1 + m_2$  is even. In this case (22) reduces to  $|T(k)|^2 + |R^r(k)R^l(k)| = 1$ , and similarly to the case of real potentials,  $|R^{l/r}|$  and  $|T|$  are bounded-above

by 1. If  $m_1 + m_2$  is odd, (22) becomes  $|T(k)|^2 - |R^r(k)R^l(k)| = 1$ ,  $|R^{l/r}|$  and  $|T|$  need not be bounded, and the potential can support spectral singularities.

It is also worth mentioning that in light of (22), a  $\mathcal{PT}$ -symmetric potential is unidirectionally or bidirectionally reflectionless at a wavenumber  $k$  if and only if the transmission coefficient  $|T(k)|^2$  is unity, i.e.,  $T(k) = e^{i\tau(k)}$ .

Another obvious consequence of Eqs. (4) and (5) is the identity:  $\mathbf{M}(-k) = \sigma_1 \mathbf{M}(k) \sigma_1$ , which means

$$M_{11}(-k) = M_{22}(k), \quad M_{12}(-k) = M_{21}(k). \quad (24)$$

Writing these equations in terms of the reflection and transmission amplitudes, we find

$$R^l(-k) = -\frac{R^r(k)}{D(k)}, \quad R^r(-k) = -\frac{R^l(k)}{D(k)}, \quad T(-k) = \frac{T(k)}{D(k)}, \quad (25)$$

where  $D(k) := T(k)^2 - R^l(k)R^r(k)$ . The following are some notable implications of these relations.

- If  $v(x)$  is a real potential, Eqs. (2) and (18) hold, and we can use them together with (25) to establish

$$D(k) = e^{2i\tau(k)} = \frac{T(k)}{T(k)^*}, \quad (26)$$

$$R^{l/r}(-k) = R^{l/r}(k)^*, \quad (27)$$

$$T(-k) = T(k)^*. \quad (28)$$

Using the first two of these equations, we can respectively express the reciprocity and unitarity relations (1) and (2) as

$$|R^l(-k)| = |R^r(k)|, \quad (29)$$

$$R^{l/r}(k)R^{l/r}(-k) + |T(k)|^2 = 1. \quad (30)$$

Clearly, (29) is identical to (7) and (28) implies (8). Therefore, Eqs. (7) and (8) that are conjectured to hold for  $\mathcal{PT}$ -symmetric potentials in Ref. [5] are indeed satisfied by real scattering potentials. Let us also note that for real potentials (7) is just an alternative expression for the reciprocity relation (1).

- If  $v(x)$  is a  $\mathcal{PT}$ -symmetric potential, Eqs. (20) – (23) hold, and it is easy to show that the last two of these relations imply (26). Using this equation in (25), we find

$$R^{l/r}(-k) = -e^{2i\tau(k)} R^{r/l}(k), \quad T(-k) = T(k)^*. \quad (31)$$

If we take the absolute-value of both sides of these equations, we arrive at Eqs. (7) and (8). This provides a proof of the conjecture that every  $\mathcal{PT}$ -symmetric scattering potential comply with (7) and (8), [5]. As we explained above, these relations also hold for the real potentials and (7) is equivalent to the reciprocity relation (1). We may, therefore, refer to it as the “reciprocity relation” also for the complex  $\mathcal{PT}$ -symmetric potentials.

Next, we use the first equation in (31) together with (17), (22), and (23) to compute:

$$\begin{aligned}
R^l(k)R^l(-k) + |T(k)|^2 &= -e^{-2i\tau(k)}R^l(k)R^r(k) + |T(k)|^2 \\
&= -e^{i[\lambda(k)+\rho(k)-2\tau(k)]}|R^l(k)R^r(k)| + |T(k)|^2 \\
&= e^{i(m_1+m_2)\pi}|R^l(k)R^r(k)| + |T(k)|^2 = 1
\end{aligned} \tag{32}$$

Following the same approach, we also find  $R^r(k)R^r(-k) + |T(k)|^2 = 1$ . This leads us to the remarkable conclusion that, similarly to (26) and (28), (30) is also a property of real scattering potentials that is shared by complex  $\mathcal{PT}$ -symmetric scattering potentials. Because for real potentials it coincides with the unitarity condition (2), we may view it as a generalized unitarity or pseudo-unitarity relation [12].

In conclusion, searching for a proof of the relations (7) and (8) and using the transformation properties of the transfer matrix of one-dimensional scattering theory, we have obtained a number of interesting identities for the reflection and transmission amplitudes of general  $\mathcal{PT}$ -symmetric scattering potentials. Eqs. (7) and (8) follow as immediate consequences of these identities and hold true also for real potentials. Perhaps more interestingly, we have found a particular form of the unitarity relation for real potentials, namely (30), that is also satisfied by  $\mathcal{PT}$ -symmetric complex potentials. These observations uncover interesting similarities between real and complex  $\mathcal{PT}$ -symmetric scattering potentials.

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